C*-ALGEBRAS

CONTEXT

The claim of this work is the definition of a pertinent C*-algebra on $\mathcal{A} = (E_N, \mathbb{C}, \triangle_{\blacktriangledown A})$. Our initial motivations for the coming developments have already been explained in [01] and are rooted in physical considerations related to a possible special representation of pure magnetic fields. The intervention of C*-algebras in theoretical physics is not really new (see, e.g., [06]) but appears to be more and more essential.

Indeed there are also other motivations as for example recalled in [05; page 3]: … "The formal structure of a generally covariant quantum theory is as follows. Let Γ_{ex} be the space of solutions of the generally covariant equations of motion endowed with a degenerate symplectic structure defined by these equations. The degenerate directions of this symplectic structure integrate in orbits and the solutions which belong to the same orbit must be physically identified. The orbits form a symplectic space Γ - a fully covariant object which becomes the physical phase space of the theory. The set A = $C^\infty(\Gamma)$ of the real smooth functions on Γ, called the physical observables, forms an Abelian multiplicative algebra. These observables are regarded as classical limits of the non-commuting quantum observables whose ensemble forms the non-Abelian C*-algebra A." …

Each term in the above citation receives a peculiar echo in our investigations. A long time ago (in the early 80teens), at the beginning just for the fun and as exercise, we decided to start the construction of an extrapolation of the now called "classical" generalized theory of relativity (GTR). The generalization lays entirely on the fact that variations until the second order of the tetrad had to be taken into account in the development. This investigation gave rise to unexpected results essentially related to the fact that we were trying to reconstruct a Riemann tensor. As it can be actually red in [07], that reconstruction is possible. It is obtained within two distinct families of situations. The first one is related to symmetric connections (as, for example, in GTR) and naturally yields a geometric symplectic structure (a Clifford algebra based on the first partial derivates of the tetrad). This was suggesting, at least as provisory conclusion, that variations of the tetrad including the second order of these variations naturally incorporate a geometric symplectic structure isomorphic to EM fields.

The investigation of the second family is still in progress. We shall come back a little bit later to that point and to the commentaries of [05; page 3].

SOME CRUCIAL CHOICES

Keeping that purpose and all these things in mind, former explorations encourage us to make the first following choices:

1) Extended (tensor) products of the theory have to be built with the help of antisymmetric cubes;

2) The function ${}_{\blacktriangledown A}\Phi : \forall a \in E_N(\mathbb{C}), {}_A\Phi(a) \in M_N(\mathbb{C}) \mid \forall b \in E_N(\mathbb{C}), {}_A\Phi(a). b = \triangle_{\blacktriangledown A}(a, b)$ should define a multiplicative morphism;

3) The adjoint element a* of any element a must be its extended squared product; in extenso: $a^* = \triangle_{\blacktriangledown A}(a, a)$.

SOME CONSEQUENCES

The first consequences of these preliminary choices are the following:

1) $\Leftrightarrow \forall a, b \in E_N(K)$, the relation $(a + b)^* = a^* + b^*$ holds; this is one of the necessary complementary condition defining a C*-algebra

$\Leftrightarrow \mathcal{A} = (E_N, \mathbb{C}, \triangle_{\blacktriangledown A})$ is a Lie algebra

⇔ This hypothesis is compatible with a theory taking into consideration the variations of the tetrad until the second order inclusively and building a tool mimicking the Riemannian tensor; in that case precisely, the cube at hand is the torsion.

2) ⇔ $\mathcal{A} = (E_N, \mathbb{C}, ⌂_{\blacktriangledown A})$ is associative

⇔ Any mathematical tool mimicking the Riemannian tensor built with the cube ▼A is reduced to its representation in the "normal coordinates" world.

In fact, although we are convinced to investigate on a pure mathematical side the construction of a pertinent C*-algebra, we are simultaneously doing physics and answering some underlying questions related to the second family we were speaking about in the former paragraph.

At this stage of the theory, we would like to make a pictorial remark that we hope to be helpful for the readers. A. Einstein's theory (the GTR) perfectly describes and explains the behavior of the front glass of a car (including motion, expansion, contraction…) whilst our modest work tries to describe not the motion of a drop of water on that front glass but the motion of inner deformations of that front glass which is perhaps expanding.

LOOKING FOR A NEUTRAL ELEMENT

The list of the above choices is not exhaustive but it doesn't matter. Let us now focus on the research of $1_{\mathcal{A}}$. This is absolutely not trivial. A first and quite more general exploration involving the quaternionic numbers has been developed in [02]. Although it was yielding a hopeful possible link between the 1 of that set and a representation of the Lorentz transformations we reduce our self here to a modest investigation on $\mathcal{A} = (E_N, \mathbb{C}, ⌂_{\blacktriangledown A})$. Precisely we want to make the following

REMARK:

"There is a link between the left $1_{\mathcal{A}^-}$ and a special feature of the problematic scrutinizing the possibility to write the Lorentz Einstein Law (LLE) under the formalism of a second order differential operator (See an introduction in [03]."

A subsidiary comment concerning that point is that the LLE is generally considered as the generic solution of the GTR equations for particles obeying both EM and gravitation fields. In that sense our own investigations are entering inside the context defined in [05; page 3].

DEMONSTRATION:

DEFINITION: LEFT-NEUTRAL ELEMENT ON …

Let us first recall the definition of a left-neutral element $1_{\mathcal{A}^-}$: $\forall\, a \in E_N(\mathbb{C})$, $⌂_{\blacktriangledown A}(1_{\mathcal{A}^-}, a) = a$.

In extenso this is: $\forall\, a \in E_N(\mathbb{C})$, $A_{\gamma\delta}{}^{\mu}.\, 1_{-}{}^{\gamma}.\, a^{\delta} = a^{\delta} \rightarrow A_{\gamma\delta}{}^{\mu}.\, 1_{-}{}^{\gamma} = \delta_{\mu}{}^{\delta}$.

THE LLE AND ITS SECOND ORDER DIFFERENTIAL OPERATOR FORMULATION

The link we want to demonstrate appears spontaneously in [03; § 1.4, pages 6-7] for a special and limited configuration of the problematic.

Provided four conditions (c-1, 2, 3 and 4; see [03]) hold the following set of equations:

$$\frac{dA^\theta}{ds} = \sum_\lambda \sum_\mu {}_\theta q_{\lambda\mu} . x^\lambda . \frac{dx^\mu}{ds} + q^\theta \; ; \; {}_\theta q_{\lambda\mu} \neq 0$$

transforms the simplified second order differential operator:

$$p_0 . \frac{d^2}{d^2 s} A^\theta + p_1 . \frac{dA^\theta}{ds} + p_2 . A^\theta = L[A^\theta(s)]$$

into an expression of the LLE. The confrontation between the first (c-1) and the second (c-2) condition yields:

$$\sum_\lambda \Gamma_{\lambda\mu}{}^\theta . x^\lambda = \delta^\theta{}_\mu$$

The conclusion is that that above set of equations is automatically associated with the existence of a left neutral element of $\mathcal{A}$ when ▼A = ▼Γ and this is a very fundamental remark for several reasons that we shall now try to explain in details.

OTHER REMARKS

Before that, we have to make the complementary but important remarks that:
1°) it can be demonstrated that the set of these equations is a set of classical translations within the space of the speeds of the particle. This is suggesting a logical link between the classical formulation of the transformations of the speeds (= non relativistic) and the left neutral mathematical element of our approach.

2°) with that way of thinking, the left neutral element is a priori not unique and, as long as the space can be projected on the axis of a basis (equivalent to the assertion: basis vector of the tetrad belong to $1_{\mathcal{A}}{}^{-}$), defines a set of invariant positive connections. In the former cases the connections are also symmetric and thus compatible with the GTR.

3°) the situation becomes quite more unclear for antisymmetric connections (in fact torsions if we work with the second family defined above: see context) and we certainly then have to envisage complicated topologies on subspaces of $E_N(\mathbb{C})$.

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