1. Equation of the conoide in a 3D right hand basis with an elastic wedge product in coincidence with the usual cross product:

In the two last sections (09 and 10), we could win the sensation that a coherence equation follows from the realization of any splitting:

$$\vec{u} \times \vec{w} = \begin{pmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{23} \\ p_{31} & p_{32} & p_{33} \end{pmatrix} . \vec{w} + \vec{z} \qquad (1.1)$$

in a 3D right hand basis where the elastic wedge product is coinciding with the classical wedge (here: cross) product, that is, in extenso [section 10 § 1.4. (2)]:

$D - ½. \sum_a \sum_b \partial_{ab}f. (\mathbf{s} - \mathbf{u})^a. (\mathbf{s} - \mathbf{u})^b$
$=$
$| P | - \sum_a \sum_b c_{ab}. s^a. s^b$ (1.2)

But we also could demonstrate that [section 10; relations (9) to (14)]:

$½. \sum_a \sum_b \partial_{ab}f. (\mathbf{s} - \mathbf{u})^a. (\mathbf{s} - \mathbf{u})^b - \sum_a \sum_b c_{ab}. s^a. s^b$
$=$
$f(\mathbf{u}) + |P|$ (1.3)

A confrontation between (1.2) and (1.3) yields:

$D - |P| = f(\mathbf{u}) + |P|$ (1.4)

Accordingly to the relation [Noemie04. Pdf; page 27]

$|S(\mathbf{w})| = f(\mathbf{u}) = c_{11}. (u^1)^2 + c_{22}. (u^2)^2 + c_{33}. (u^3)^2 + c_{12}. u^1. u^2 + c_{23}. u^2. u^3 + c_{13}. u^1. u^3 + c_1. u^1 + c_2. u^2 + c_3. u^3 - |P|$

where $S(\mathbf{w})$ is in fact corresponding to the linear system that one gets in writing (1.1) under the following form:

$S(\mathbf{w}). \mathbf{w} = \mathbf{z}$

One must now absolutely remark that:

$\mathbf{u} \times \mathbf{w} - [P]. \mathbf{w} = \mathbf{z}$

$\Phi(\mathbf{u}). \mathbf{w} - [P]. \mathbf{w} = \mathbf{z}$

$\{\Phi(\mathbf{u}) - [P]\}. \mathbf{W} = \mathbf{z}$

$S(\mathbf{w}) = \Phi(\mathbf{u}) - [P]$ (1.5)

this is resulting in:

$|S(\mathbf{w})| = |\Phi(\mathbf{u}) - [P]|$ (1.6)

and specially here in a 3D space:

$|S(\mathbf{w})| = |(-1). \{[P] - \Phi(\mathbf{u})\}| = (-1)^3. |[P] - \Phi(\mathbf{u})|$ (1.7)

In 3D basis where we are dealing we finally get:

$|S(\mathbf{w})| = - D$ (1.8)

and:

$f(\mathbf{u}) + D = 0$ (1.9)

This is an important result for the 3D right hand basis with an ewp in coincidence with the cross product. This also leads, in confrontation with (1.4), to another surprising result:

$D = |P| = |[P] - \Phi(\mathbf{u})|$ (1.10)

This result means that trivial splitting only exist in a 3D right hand basis where the elastic wedge product is coinciding with the classical cross product under the condition:

$f(\mathbf{u}) = D = |P| = 0$ (1.11)

equivalent to:

$c_{11}. (u^1)^2 + c_{22}. (u^2)^2 + c_{33}. (u^3)^2 + c_{12}. u^1. u^2 + c_{23}. u^2. u^3 + c_{13}. u^1. u^3 + c_1. u^1 + c_2. u^2 + c_3. u^3 = 0$ (1.12)

Because of the hypothesis concerning the basis and the coincidence between the elastic wedge product and the classical wedge product:

$[A] = [J]$ (1.13)

$|A| = - 1$ (1.14)

with [A] :

$$\begin{matrix} A_{12}^{1} & A_{12}^{2} & A_{12}^{3} \\ A_{23}^{1} & A_{23}^{2} & A_{23}^{3} \\ A_{13}^{1} & A_{13}^{2} & A_{13}^{3} \end{matrix}$$

and:

$$[J] = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} \qquad (1.15)$$

and because all coefficients of the conoide f(**u**) are given by the relations (15.1 to 15.10) of the section 10 (blue color to help):

$c_{11} = p_{11}. (A_{12}^{3}. A_{13}^{2} - A_{12}^{2}. A_{13}^{3})$
$+ p_{21}. (A_{12}^{1}. A_{13}^{3} - A_{12}^{3}. A_{13}^{1})$
$+ p_{31}. (A_{12}^{2}. A_{13}^{1} - A_{12}^{1}. A_{13}^{2})$ (15-1)

$c_{22} = p_{12}. (A_{12}^{3}. A_{23}^{2} - A_{12}^{2}. A_{23}^{3})$
$+ p_{22}. (A_{12}^{1}. A_{23}^{3} - A_{12}^{3}. A_{23}^{1})$
$+ p_{32}. (A_{12}^{2}. A_{23}^{1} - A_{12}^{1}. A_{23}^{2})$ (15-2)

$c_{33} = p_{13}. (A_{13}^{3}. A_{23}^{2} - A_{13}^{2}. A_{23}^{3})$
$+ p_{23}. (A_{13}^{1}. A_{23}^{3} - A_{13}^{3}. A_{23}^{1})$
$+ p_{33}. (A_{13}^{2}. A_{23}^{1} - A_{13}^{1}. A_{23}^{2})$ (15-3)

$c_{12} = p_{11}. (A_{12}^{3}. A_{23}^{2} - A_{12}^{2}. A_{23}^{3})$
$+ p_{12}. (A_{13}^{2}. A_{12}^{3} - A_{13}^{3}. A_{12}^{2})$
$+ p_{21}. (A_{12}^{1}. A_{23}^{3} - A_{12}^{3}. A_{23}^{1})$
$+ p_{22}. (A_{13}^{3}. A_{12}^{3} - A_{13}^{1}. A_{12}^{3})$
$+ p_{31}. (A_{12}^{2}. A_{23}^{1} - A_{12}^{1}. A_{23}^{2})$
$+ p_{32}. (A_{13}^{1}. A_{12}^{2} - A_{13}^{2}. A_{12}^{1})$ (15-4)

$c_{13} = p_{11}. (A_{13}^{2}. A_{23}^{3} - A_{23}^{2}. A_{13}^{3})$
$+ p_{13}. (A_{12}^{3}. A_{13}^{2} - A_{12}^{2}. A_{13}^{3})$
$+ p_{21}. (A_{13}^{3}. A_{23}^{1} - A_{13}^{1}. A_{23}^{3})$
$+ p_{23}. (A_{12}^{1}. A_{13}^{3} - A_{12}^{3}. A_{13}^{1})$
$+ p_{31}. (A_{13}^{1}. A_{23}^{2} - A_{13}^{2}. A_{23}^{1})$
$+ p_{33}. (A_{12}^{2}. A_{13}^{3} - A_{12}^{1}. A_{13}^{2})$ (15-5)

$c_{23} = p_{12}. (A_{23}^{2}. A_{13}^{3} - A_{23}^{3}. A_{13}^{2})$
$+ p_{13}. (A_{12}^{3}. A_{23}^{2} - A_{12}^{2}. A_{23}^{3})$
$+ p_{22}. (A_{23}^{3}. A_{13}^{1} - A_{23}^{1}. A_{13}^{3})$
$+ p_{23}. (A_{12}^{1}. A_{23}^{3} - A_{12}^{3}. A_{23}^{1})$
$+ p_{32}. (A_{23}^{1}. A_{13}^{2} - A_{23}^{2}. A_{13}^{1})$
$+ p_{33}. (A_{12}^{2}. A_{23}^{3} - A_{12}^{3}. A_{23}^{2})$ (15-6)

$$c_1 = A_{12}^{1}.\,(p_{31}.\,p_{23} - p_{21}.\,p_{33}) + A_{12}^{2}.\,(p_{33}.\,p_{11} - p_{31}.\,p_{13}) + A_{12}^{3}.\,(p_{21}.\,p_{13} - p_{11}.\,p_{23}) + A_{13}^{1}.\,(p_{21}.\,p_{32} - p_{22}.\,p_{31}) + A_{13}^{2}.\,(p_{31}.\,p_{12} - p_{11}.\,p_{32}) + A_{13}^{3}.\,(p_{11}.\,p_{22} - p_{21}.\,p_{12}) \quad (15\text{-}7)$$

$$c_2 = A_{12}^{1}.\,(p_{32}.\,p_{23} - p_{22}.\,p_{33}) + A_{12}^{2}.\,(p_{12}.\,p_{33} - p_{32}.\,p_{13}) + A_{12}^{3}.\,(p_{13}.\,p_{22} - p_{12}.\,p_{23}) + A_{23}^{1}.\,(p_{21}.\,p_{32} - p_{31}.\,p_{22}) + A_{23}^{2}.\,(p_{31}.\,p_{12} - p_{32}.\,p_{11}) + A_{23}^{3}.\,(p_{22}.\,p_{11} - p_{21}.\,p_{12}) \quad (15\text{-}8)$$

$$c_3 = A_{13}^{1}.\,(p_{32}.\,p_{23} - p_{22}.\,p_{33}) + A_{13}^{2}.\,(p_{12}.\,p_{33} - p_{32}.\,p_{13}) + A_{13}^{3}.\,(p_{13}.\,p_{22} - p_{12}.\,p_{23}) + A_{23}^{1}.\,(p_{21}.\,p_{33} - p_{31}.\,p_{23}) + A_{23}^{2}.\,(p_{31}.\,p_{13} - p_{33}.\,p_{11}) + A_{23}^{3}.\,(p_{23}.\,p_{11} - p_{21}.\,p_{13}) \quad (15\text{-}9)$$

$$c = -\,|P| \quad (15\text{-}10)$$

this is specially resulting here in:

$c_{11} = -\,p_{11}$ (1,16)
$c_{22} = -\,p_{22}$ (1.17)
$c_{33} = -\,p_{33}$ (1.18)
$c_{12} = -\,(p_{12} + p_{21})$ (1.19)
$c_{13} = -\,(p_{13} + p_{31})$ (1.20)
$c_{23} = -\,(p_{23} + p_{32})$ (1.21)
$c_1 = (p_{21}.\,p_{13} - p_{31}.\,p_{12}) + p_{11}.\,(p_{32} - p_{23})$ (1.22)
$c_2 = (p_{13} - p_{31}).\,p_{22} + (p_{21}.\,p_{32} - p_{12}.\,p_{23})$ (1.23)
$c_3 = (p_{32}.\,p_{13} - p_{31}.\,p_{23}) + (p_{21} - p_{12}).\,p_{33}$ (1.24)
$c = -\,|P|$ (1.25)

2. Case of a symmetric matrix [P]:
One must remark that in a case of a symmetric matrix [P], linear coefficients of f(**u**) disappear . Indeed:

$$\forall\ k \text{ and } \forall\ m = 1, 2 \text{ or } 3,\ p_{km} - p_{mk} = 0 \quad (1.26)$$
$$\Downarrow$$
$$\forall\ k = 1, 2 \text{ or } 3,\ c_k = 0 \ (1.27)$$
$$\Downarrow$$
$$f(\mathbf{u}) = c_{11}.\,(u^1)^2 + c_{22}.\,(u^2)^2 + c_{33}.\,(u^3)^2 + c_{12}.\,u^1.\,u^2 + c_{23}.\,u^2.\,u^3 + c_{13}.\,u^1.\,u^3 - |P| \ (1.28)$$
$$\Downarrow$$
$$f(\mathbf{u}) = -\,[p_{11}.\,(u^1)^2 + p_{22}.\,(u^2)^2 + p_{33}.\,(u^3)^2 + 2.\,(p_{12}.\,u^1.\,u^2 + p_{23}.\,u^2.\,u^3 + p_{13}.\,u^1.\,u^3) + |P|\,] \quad (1.29)$$

Where one immediately remarks that:

$f(\mathbf{u}) = f(-\,\mathbf{u})$ (1.30)
$f(\mathbf{0}) = -\,|P|$ (1.31)
$\Delta f(\mathbf{u}) = f(\mathbf{u}) - f(\mathbf{0}) = p_{k\,m}.\,u^k.\,u^m$ appears to be a quadratic form depending on the components of **u**.

3. Case of a symmetric [P] with Jacobi's formalism:
Furthermore, if the matrix [P] would have both, the Jacobi's formalism $T_2(o)(\partial_{\mathbf{u}}, \partial_{\mathbf{u}} f(\mathbf{u}))$ and a symmetric formalism, equivalent to:

$$\{p_{k\,m} - \partial_{k\,m} f = 0,\ \partial_{k\,m} f - \partial_{m\,k} f = 0\} \quad (1.32)$$

this would give to f(**u**) a very special formulation, in extenso:

$$f(\mathbf{u}) = -\,[\partial_{11} f.\,(u^1)^2 + \partial_{22} f.\,(u^2)^2 + \partial_{33} f.\,(u^3)^2 + 2.\,(\partial_{12} f.\,u^1.\,u^2 + \partial_{23} f.\,u^2.\,u^3 + \partial_{13} f.\,u^1.\,u^3) + |P|\,] \quad (1.33)$$

Note that:

$$\partial_{11} f.\,(u^1)^2 + \partial_{22} f.\,(u^2)^2 + \partial_{33} f.\,(u^3)^2 + 2.\,(\partial_{12} f.\,u^1.\,u^2 + \partial_{23} f.\,u^2.\,u^3 + \partial_{13} f.\,u^1.\,u^3) = \langle \mathbf{u} \mid T_2(o)(\partial_{\mathbf{u}}, \partial_{\mathbf{u}} f(\mathbf{u})).\,\mathbf{u} \rangle \quad (1.34)$$

One could also write (with the "inner product" or with the "bracket" notation):

$$f(\mathbf{u}) = -\,\langle \mathbf{u} \mid T_2(o)(\partial_{\mathbf{u}}, \partial_{\mathbf{u}} f(\mathbf{u})).\,\mathbf{u} \rangle + f(\mathbf{0}) \quad (1.35)$$

Suppose now that we could have made a Taylorisation of this function f around zero, we would have consequently written, respecting usual rules:

$$f(\mathbf{0} = \mathbf{u} - \mathbf{u}) = f(\mathbf{u}) + \partial_k f(\mathbf{u}).\,(-\,u^k) + ½.\,\partial_{k\,m} f(\mathbf{u}).\,u^k.\,u^m + \ldots \quad (1.36)$$

But (1.22) implies:

$$\mathbf{c} = \mathbf{0} \Rightarrow \partial_{\mathbf{u}} f(\mathbf{u}) = T_2(o)(\partial_{\mathbf{u}}, \partial_{\mathbf{u}} f(\mathbf{u})).\,\mathbf{u} \quad (1.37)$$

Because of (1.28):

$$½.\,\partial_{k\,m} f(\mathbf{u}).\,u^k.\,u^m = ½.\,\langle \mathbf{u} \mid T_2(o)(\partial_{\mathbf{u}}, \partial_{\mathbf{u}} f(\mathbf{u})).\,\mathbf{u} \rangle \quad (1.38)$$

Thus the Taylorisation would have been:

$$f(\mathbf{0}) = f(\mathbf{u}) - \langle \mathbf{u} \mid \partial_{\mathbf{u}} f(\mathbf{u}) \rangle + ½.\,\partial_{k\,m} f(\mathbf{u}).\,u^k.\,u^m + \ldots$$

$$f(\mathbf{0}) = f(\mathbf{u}) - \langle \mathbf{u} \mid T_2(o)(\partial_{\mathbf{u}}, \partial_{\mathbf{u}} f(\mathbf{u})).\,\mathbf{u} \rangle + ½.\,\langle \mathbf{u} \mid T_2(o)(\partial_{\mathbf{u}}, \partial_{\mathbf{u}} f(\mathbf{u})).\,\mathbf{u} \rangle + \ldots$$

$$f(\mathbf{0}) = f(\mathbf{u}) - ½.\,\langle \mathbf{u} \mid T_2(o)(\partial_{\mathbf{u}}, \partial_{\mathbf{u}} f(\mathbf{u})).\,\mathbf{u} \rangle + \ldots \quad (1.39)$$

A confrontation between (1.35) and (1.39) yields an interesting mathematical problem:

$$f(\mathbf{u}) - f(\mathbf{0}) = ½.\,\langle \mathbf{u} \mid T_2(o)(\partial_{\mathbf{u}}, \partial_{\mathbf{u}} f(\mathbf{u})).\,\mathbf{u} \rangle + \ldots \quad (1.40)$$

because it is obviously in contradiction with (1.29) and indirectly implying that missing terms in the Taylorisation (1.36) represent a subsequent quantity absolutely not negligible.

4. Commentaries:
From all these results we want to point out that the symmetry of the matrix [P] is a sufficient condition for Δf(**u**) to be a quadratic form. This fact is linking our thoughts to an interrogation which is to determinate when and how this quadratic form could be connect to the local metric of the 3D right hand basis where the elastic wedge product is coinciding with the classical wedge product and where we are realizing any splitting with the (1.1) formalism.

That is the reason why we proposed to interpret $\sum_a \sum_b c_{ab}.\,s^a.\,s^b$ (when **u = s**; see the end of section 10) as the trajectory of a physical phenomenon associated with [P]. **u = 0** and **IO = 0**, that is in our mind, without spin. The 3D approach must only help to precise the relations between metric ($[c_{ab}]$), equation of motion ([P]. **u = 0** or $0 = \sum_a \sum_b c_{ab}.\,s^a.\,s^b$) and spin (**IO = 0**). We think that it is a good prelude to the description of the photon in a 4D space and may be, after that, to any particle in such space.

Leaving for a while the problematic involving the paths through the time (Feynman's integrals) and staying in a 3D space where a lot of things occur instantaneously or, better said, at each instant of the chronology we are living in, we could test our proposition for bodies or for particles with a spin, i. e. with a vector Io not equal to zero.

If we now come back to the Lorentz-Einstein Law in 4D space and its Kern Law obtained in an embedded 3D space with the help of the RDT (Russian Dolls Technique), we remember that non trivial splitting are accompanied by a non zero rest which is a relative acceleration. Non trivial splitting are also associated with a Δ and a D ≠ 0 and given by [section 09; (15)]: $D = ⅛.\,\Delta + ½.\,\sum_a \sum_b \partial_{ab} f.\,(\mathbf{IO})^a.\,(\mathbf{IO})^b$. Thus, in our way of thinking, non trivial splitting are related to an accelerated motion with or without spin (and in this case: D =

$\frac{1}{8}$. Δ). The formalism of D - $\frac{1}{8}$. Δ is similar to the Hamiltonian that would describe an interaction of the spin **IO** with itself [39], called it H_s. In the absence of spin one gets the mathematical evidence D - $\frac{1}{8}$. $\Delta = 0$ which is consequently associated with $H_s = 0$.

Knowing the subtle difference between a geodesic and an auto-parallel, as clearly explained in [41; pages 1 to 10], we want to come back to our initial intuition in the way to make an intelligent use of the elastic wedge product in physics.

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