Abstract

We propose to re-examine the properties of the four dimensional speed vector defined within the generalized theory of relativity to built a link between our mathematical investigations and some physical phenomenon actually under experiment, namely: the Thirring Lense effect and the geometric magnetism (Berry's phase). We first start with the evidence that the invariant scalar characterizing a given solution of the law of motion of a relativistic particle can always be written under the formalism of a conic in the embedded three dimensional space. We then define the singular point of this conic which in turn appears to be in fact an infinitesimal zone in the space of speeds. We recall the natural link between the notion of extended product and any conic. We finally apply these thoughts to progressing plane waves in vacuum for which we make the remark that a trivial split of the extended product they are involved with only occurs in average or at the limit but is never exactly realized because associated with a mathematical incoherence or with a degenerate metric. If we try to explain this remark in saying that any plane wave can be assimilate to an elementary particle that we propose to describe via an energetic flow of invariant volumetric density of energy, we arrive to the formal prediction that any modification of the metric should induce a modified magnetic field for plane waves progressing in this space time. We investigate the special case of the Thirring-Lense effect around the earth and come to the conclusion that our prediction yields a very weak modification of the magnetic field intensity. We relate our approach to another one concerning the study of the Berry's phase.

Within the relativistic approach, any particle follows geodesics. It implies that:

(1)

$$^{(4)}\mathbf{v}^2 = k$$

where k is a real constant and $^{(4)}\mathbf{v}$ is the four dimensional speed vector of the particle under consideration, always is a solution of the law of motion for a given particle [01; page 107; exercise (20.1)]. Remark that a vanishing real constant (k = 0) transforms the four dimensional speed vector of the particle into an isotropic vector [02; page 3] and this could be the starting point for a widest and more modern approach involving the notion of spinor. For now, let us suppose that we work on a part of a space $(E_4, \mathbb{R})$ where a metric is defined everywhere by a (4-4) matrix:

(2)

$$^{(4)}[G] = T_2(.)(\Omega, \Omega) = [\ldots , \mathbf{e}_\alpha . \mathbf{e}_\beta, \ldots]$$

where $\Omega \equiv (\ldots , \mathbf{e}_\alpha, \ldots)$ is any element of $(E_4, \mathbb{R})^4$ with the property to be a basis for $(E_4, \mathbb{R})$. In these conditions, the relation (1) can also be written:

(3)

$$< \mathbf{v}|. ^{(4)}[G]. |\mathbf{v}> = g_{\alpha\beta}. v^\alpha. v^\beta = 0$$

To go further in this work one should read etgb09.pdf, etgb10.pdf, etgb11.pdf and Annexe B. pdf for the verifications because we shall make use of the results obtained in these documents without making any new demonstration. This being done, we shall remark that (3) can always be decomposed in:

(4)

$$g_{ii}. (v^i)^2 + \sum_{i<j} g_{ij}. v^i. v^j + \sum_{i<j} g_{ji}. v^j. v^i + \sum_i g_{0i}. v^0. v^i + \sum_i g_{i0}. v^i. v^0 + g_{00}. (v^0)^2 = k$$

which obviously, because $\mathbb{R}$ is equipped with a commutative product, can also be written:

(5)

$$g_{ii}. (v^i)^2 + 2. \sum_{i<j} g_{ij}. v^i. v^j + 2. \sum_i g_{0i}. v^0. v^i + g_{00}. (v^0)^2 - k = 0$$

and this relation owns the characteristic formalism of a vanishing 3D conic:

(6)

$$C(\ldots , v^i, \ldots) = C(\mathbf{v}) = c_{ii}. (v^i)^2 + \sum_{i<j} c_{ij}. v^i. v^j + \sum_i c_i. v^i + c = 0$$

if we define the correct correspondences:

(7)

$$c_{ii} = g_{ii}$$
$$c_{ij} = 2. g_{ij}$$
$$c_i = 2. g_{0i}. v^0$$
$$c = g_{00}. (v^0)^2 - k$$

Let us suppose in a first exploration that the components of the local metric do not depend on the speed of the phenomenon under study. This means that the speed of this object does not change the geometry even if it does not avoid variations of the metric from a place to the other; but for some other reason. This hypothesis being accepted, we get:

(8)

$$\partial C(\mathbf{v})/\partial v^k = 2.\ c_{kk}.\ v^k + \sum_{i \neq k} c_{ik}.\ v^i + c_k$$

and with the ad hoc correspondences (7):

(9)

$$\partial C(\mathbf{v})/\partial v^k = 2.\ (\sum_i\ g_{ik}.\ v^i + g_{0k}.\ v^0)$$

In a vector language, this can be written:

(10)

$$\partial_v C({}^{(3)}\mathbf{v}) = 2.\ \{{}^{(3)}[G].\ {}^{(3)}\mathbf{v} + v^0.\ {}^{(3)}\mathbf{g}\}$$

where ${}^{(3)}[G]$ is the (3-3) matrix representing the spatial part of the metric tensor, ${}^{(3)}\mathbf{g}$ is the ordinary vector with coordinates $(\ldots, g_{0k}, \ldots)$. It has been clearly shown that 3D conics can be classified into two families: the proper conics and the other one. Conics of interest for us are proper conics and own a singular vector ${}^{(3)}\mathbf{s}$. The latter exists here if we can find a solution for the system:

(11)

$${}^{(3)}[G].\ {}^{(3)}\mathbf{s} + v^0.\ {}^{(3)}\mathbf{g} = \mathbf{0}$$

There is no difficulty to understand that a singular speed vector exists for $C({}^{(3)}\mathbf{v})$ if the spatial part of the metric tensor is represented by an invertible element of M(3 x 3, R), that is by an element of GL(3 x 3, R). For the algebra it is equivalent to the condition:

(12)

$$|\ {}^{(3)}[G]\ | \neq 0$$

The solution, if it exists is unique and depends not only on the metric but also on ${}^{(3)}\mathbf{g}$ and of course on v^0. Let us remember at this place that it can be useful to introduce the following notation:

(13)

$${}^{(4)}[G] = \begin{bmatrix} g_{00} & {}^{(3)}\mathbf{g} \\ {}^{(3)}\mathbf{g} & {}^{(3)}[G] \end{bmatrix}$$

because it will help us later to work with a A.D.M. formalism [03]. Up to now we shall suppose that the conic $C({}^{(3)}\mathbf{v})$ under study here is a proper conic. We still have demonstrated that such conics can be mathematically involved into the split of an extended product of the following family:

(14)

$$\triangle_{(\blacktriangledown A)}\ ({}^{(3)}\mathbf{v},\ {}^{(3)}\mathbf{w}) = {}^{(3)}[P].\ {}^{(3)}\mathbf{w} + {}^{(3)}\mathbf{K},\ \forall\ {}^{(3)}\mathbf{w} \in (E_4, R)$$

This is a very interesting family because it can be connected to the other one which is an extension of the angular momentum:

$$\triangle_{(\blacktriangledown A)}\ ({}^{(3)}\mathbf{p},\ {}^{(3)}\mathbf{x}) = {}^{(3)}[P].\ {}^{(3)}\mathbf{x} + {}^{(3)}\mathbf{K}$$

$${}^{(3)}\mathbf{p} = m.\ {}^{(3)}\mathbf{v}$$

where m is the mass of the phenomenon satisfying (1). We also have been able to reduce the cube A defining the extended product locally into a (3-3) table in some special circumstances and to prove that in such circumstances:

(15)

$$[P] = \tfrac{1}{2}.\ |A|.\ \{[A]^t.\ [J]\}.\ T_2(o)(\partial_v, \partial_v\, C(\mathbf{v})) + \{[A]^t.\ [J]\}.\ \Phi(\mathbf{s})$$

with:

(16)

$$|A| = \pm\ 1, \text{ and}$$

(17)

$$[J] = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

where **s** is the singular speed vector of C(**v**). Today, we also know that:
(18)

$$\mathbf{s} = \mathbf{IO} + \mathbf{v}$$

with:
(19)

$$\mathbf{IO} = -\, T^{-1}{}_2(o)(\partial_v, \partial_v C(\mathbf{v})).\ \partial_v C(\mathbf{v})$$

In the special case under study here, starting from (10), this is yielding the remarkable relations:
(20)

$$½.\ T_2(o)(\partial_v, \partial_v C(\mathbf{v})) = {}^{(3)}[G]$$

and:
(21)

$$\mathbf{IO} = -\, {}^{(3)}[G]^{-1}\ .\ \{{}^{(3)}[G].\ {}^{(3)}\mathbf{v} + v^0.\ {}^{(3)}\mathbf{g}) = -\,({}^{(3)}\mathbf{v} + v^0.\ {}^{(3)}[G]^{-1}\,{}^{(3)}\mathbf{g})$$

By side we can effectively verify that:
(22)

$$\mathbf{s} = -\, v^0.\ {}^{(3)}[G]^{-1}\,{}^{(3)}\mathbf{g}$$

Let us now apply all these considerations to any plane wave in vacuum. Such a wave is supposed to go straightforward with an ordinary speed vector ${}^{(3)}\mathbf{v}$ of invariant intensity c. But more generally, since we know that matter interacts with the light and conversely, it is also supposed to be able to follow geodesics in (E_4, $\mathbb{R}$). We make use of this experimental evidence to argue that any plane wave satisfies at any moment to the equation of a conic C(${}^{(3)}\mathbf{v}$) like those given above by (6) and (7).

The motion of a not interacting plane wave (if any exists) progressing in the **v** direction at invariant speed is governed by a relation like [04; page 364]:
(23)

$$\mathbf{B} - \mathbf{v} \wedge \mathbf{E} = \mathbf{0}$$

The cross vector product "∧" corresponds to the special limit case where [A] = [J] and |A| = -1 inside of our mathematical approach. In this work, it allows us to propose the existence of a more general relation like:
(24)

$$\mathbf{B} = [P].\ \mathbf{E} + \mathbf{T}$$

where [P] is defined via (15), (16) and (17) above and of which (23) is a representation at the limit. Precisely:
(25)

$$[P] = -½.\ T_2(o)(\partial_v, \partial_v C(\mathbf{v})) + \Phi(\mathbf{s})$$

This general solution obviously differs from the trivial one which is:
(26)

$$\mathbf{B} = \Phi(\mathbf{v}).\ \mathbf{E}$$

But since:
(27)

$$[P] = -{}^{(3)}[G] + \Phi(\mathbf{v} + \mathbf{IO}) = -{}^{(3)}[G] + \Phi(\mathbf{v}) + \Phi(\mathbf{IO}),$$

the trivial formulation seems to be able to be satisfied again if:
(28)

$$-{}^{(3)}[G] + \Phi(\mathbf{IO}) = [0]$$

This kind of relation should act like a warning signal for the mathematician because it is the vanishing sum of a symmetric matrix, i.e. the metric tensor, and of a skew-symmetric matrix. Since any matrix vanishes if and only

if all its components vanish, this relation is unfortunately physically meaningless, unless we decide to understand it like a limit case or only realized in average:
(29)

$$< -^{(3)}[G] + \Phi(\mathbf{IO}) > = [0]$$

This is exactly what we shall do now. The introduction of the notion of extended product help us to understand that the linear progression of a plane wave only is an approximation of the reality. In fact, any wave follows a geodesic in a more or less curved space time; the curvature being due to some local or distant interactions with external sources.

Within a "classical" quantum approach one accepts a natural duality between waves and particles. Let us make the hypothesis that any plane wave is the dual representation of a particle with an invariant volumetric density of energy ρ. That is we suppose that:
(30)

$$d\rho = 0,$$

without rejecting partial variations of the volumetric density of energy along the different axis of the basis Ω to which (E_4, $\mathbb{R}$) is referred. In extenso and up to the terms of the third order:
(31)

$$\partial_\beta \rho.\ dx^\beta + ½.\ \partial^2{}_{\beta\gamma} \rho.\ dx^\beta.\ dx^\gamma + 0(3) = 0$$

With this hypothesis, it can be demonstrated that the conservation of the stress-energy tensor of any particle yields:
(32)

$$\{(\sigma/c).\ [F] + ½.\ c.\ \{\partial^2{}_{\beta\gamma} \rho.\ v^\beta.\ dx^\gamma\}.\ ^{(4)}[G]\}.\ ^{(4)}\mathbf{v} = \mathbf{0}$$

where σ is the volumetric density of electrical charge and for which a natural three dimensional "kern" representation is:
(33)

$$\{(\sigma/c).\ \Phi(^{(3)}\mathbf{B}) + ½.\ c.\ dt.\ \{\partial^2{}_{\beta\gamma} \rho.\ v^\beta.\ v^\gamma\}.\ ^{(3)}[G]\}.\ ^{(3)}\mathbf{v} = \mathbf{0}$$

Once more time, the matrix in front of the ordinary speed vector owns exactly the same problematic formalism than the matrix (28) that should give us the trivial split of the cross vector product (23) again. The advantage of the formulation (32) is that it is a quite more general one and that it does not impose the nullity for the problematic matrix. Indeed, it is sufficient that a relation like:
(34)

$$|(\sigma/c).\ [F] + ½.\ c.\ \{\partial^2{}_{\beta\gamma} \rho.\ v^\beta.\ dx^\gamma\}.\ ^{(4)}[G\ ]| = 0$$

holds. Otherwise, if we would be obliged to write (for any EM field, and for any metric):
(35)

$$(\sigma/c).\ [F] + ½.\ c.\ \{\partial^2{}_{\beta\gamma} \rho.\ v^\beta.\ dx^\gamma\}.\ ^{(4)}[G] = [0],$$

we would also be obliged to add the following necessary conditions:
(36)

$$\sigma = 0$$

symbolizing the absence of volumetric density of electrical charge and:
(37)

$$\partial^2{}_{\beta\gamma} \rho.\ v^\beta.\ v^\gamma = 0$$

symbolizing in someway that the partial derivates of second order of the volumetric density of energy play the role normally devoted to the components of the metric. We shall study this configuration in another work.

Although one can not go from (32) to (33) in a such naïve way, our approach suggests that if any correlation between the two representations exists, then a *first and rough* identification between the LHT of (28) and the problematic matrix of (33) can be proposed. With the convention $\varepsilon = \partial^2{}_{\beta\gamma} \rho.\ v^\beta.\ v^\gamma$ we get:
(38)

$$-\ (2.\ \sigma / c^2).\ (1 / \varepsilon.\ dt).\ \Phi(^{(3)}\mathbf{B}) \equiv \Phi(\mathbf{IO})$$

This relation implies:
(39)

$$^{(3)}\mathbf{IO} = -(2.\ \sigma / c^2).\ (1 / \varepsilon.\ dt).\ ^{(3)}\mathbf{B}$$

Since it has been demonstrated in the mathematical part of our work that the 3D-"io" vector should own the characteristic of a rotational vector, it is coherent to state that this vector is proportional to a magnetic field inside this approach. Indeed, we know that the magnetic field is the rotational of the EM potential four vector $^{(4)}\mathbf{A}$.

Accordingly to the relation (21) above:
(40)

$$(^{(3)}\mathbf{v} + v^0.\ ^{(3)}[G]^{-1}\ ^{(3)}\mathbf{g}) = (2.\ \sigma / c^2).\ (1 / \varepsilon.\ dt).\ ^{(3)}\mathbf{B}$$

In a frame where $^{(4)}\mathbf{v} = (c,\ ^{(3)}\mathbf{0})$ this is yielding:
(41)

$$c.\ ^{(3)}[G]^{-1}\ ^{(3)}\mathbf{g} = (2.\ \sigma / c^2).\ (1 / \partial^2_{00}\,\rho.\ c^2.\ dt).\ ^{(3)}\mathbf{B}$$

In our mind, it is a very fascinating relation. Since the ordinary $^{(3)}\mathbf{g}$ vector can be related to the imprint left by the angular momentum of a weakly gravitating system [03; § 19.2; page 450] and sometimes identified with the geometric field induced by a Thirring Lense effect around the earth [01 ; page 166 ; relation (30.12)] :
(42)

$$^{(3)}\mathbf{g}(^{(3)}\mathbf{r}) \equiv (\ldots, g_{0k}, \ldots) = (\ldots, \eta_{0k}, \ldots) + (\ldots, h_{0k}, \ldots) \equiv {}^{(3)}\mathbf{0} + {}^{(3)}\mathbf{h}(^{(3)}\mathbf{r}) = -(4.\ \mathcal{G}.\ R_{earth}/5.\ c^3.\ r^3).\ \boldsymbol{\omega} \wedge \mathbf{r}$$

where $^{(3)}\mathbf{r}$ is the position vector relatively to the center of rotation of the earth, $\mathcal{G}$ is the universal constant of gravitation, R_{earth} is the size of the earth and $\boldsymbol{\omega}$ the ordinary angular speed vector of the earth (2π in a day), this is correlatively clearly suggesting that the Thirring Lense effect induces modifications of the magnetic field of plane waves progressing around the earth. If such an effect exists, it can be de facto related to the actual and modern problematic concerning the analysis of the Berry's phases problem.

To get an approximation of this modification, we shall incorporate very roughly the volumes into the discussion:
(43)

$$^{(3)}[G]^{-1}\ ^{(3)}\mathbf{g} = (2.\ e / c^3).\ (1 / \partial^2_{00}\text{Energy}.\ dt).\ ^{(3)}\mathbf{B}$$

and suppose that, around the earth:
(44)

$$^{(3)}[G] \approx I$$

$$^{(3)}\mathbf{g} = (2.\ e / c^3).\ (1 / \partial^2_{00}\text{Energy}.\ dt).\ ^{(3)}\mathbf{B}$$

Accordingly to the fact that modifications along the time of the energy of this "wave-particle" should be quantized, we shall also suppose that it is convenient to write these modifications as a multiple of h (Planck's constant) times the frequency of this wave. At the end and only for the seek to find a first value for the modified intensity of the magnetic field, we propose:
(45)

$$(\partial^2_{00}\ \text{energy}).\ dt \approx h/4\pi$$

This will give us:
(46)

$$^{(3)}\mathbf{g} = (8\pi.\ e / h.\ c^3).\ ^{(3)}\mathbf{B}$$

Around the Earth:
(47)

$$-(4.\ \mathcal{G}.\ R_{earth}/5.\ c^3.\ r^3).\ \boldsymbol{\omega} \wedge \mathbf{r} = (8\pi.\ e / h.\ c^3).\ ^{(3)}\mathbf{B}$$

and at the surface of the Earth ($r \to R_{earth}$):
(48)

$$B \to (2.\ h.\ \mathcal{G} / 5.\ e.\ R_{earth}.\ \text{Time of a day}) \approx 5{,}15.\ 10^{-30}\ \text{Tesla}$$

We know that it is a very bad approximation but it gives us a first idea and the purpose of this work was mainly to point out the possible connections between our theoretical mathematical investigations and some concrete applications for physics.

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