0. Overview:

Based on the tensor calculus we define an extended product on a four dimensional Banach space and, generalizing a basic scholar principle, we explore the set of all possible splits of these products. A technique, essentially built on a comparison with the projection of an Hessian, is developed and it allows the discovery of a special family of splits minimizing the scalars that can be associated with. This technique is then applied to the customary expression of the Lorentz Einstein law and yields three relations of coherence. Since the scalar associated with this law is a power (force multiplied by speed) and since the procedure is finally connecting this scalar linearly with the projection of the Hessian of the wave function of the particles under study in a way directly involving the local geometric connection, we predict that our toy-model is a good theoretical tool to scrutinize situations with a "Einstein" metric. It could yield the masses (energies) for the particles concerned by these situations.

This work is a shorted version of the complete section etf31v4.pdf written in French language.

1. Introduction:

One could ask why I am giving so much importance to the notion of extended products. Indeed they only are some specific products of the tensor calculus. One reason for my obsession comes from physics; in reality from the fact that any expression of the Lorentz Einstein Law (LEL) can be seen as *an extended square product that has been split in a precise manner accordingly to some local rules connected with the local topology. Let us consider a manifold M with a Riemannian metric g, it is a priori absolutely not evident that a LEL must be satisfied at a given p of M. But since the GTR has been discovered and since it is well accepted, we know that if a LEL holds at p, then a particle (or something equivalent to) under the double influence of a gravitation and of an EM field must be in p. The LEL explains, contains implicitly the ratio between natural quantities like on one side the mass, the charge of the particle and the two fields on the other side. It also contains implicitly the information that the EM field push this particle in p out of its natural path; in extenso: the path that it would have had in absence of EM field.
In fact the analysis of this LEL is quite more complicated. Why? Firstly because we actually know that some topological changes can induce EM fields; thus, even neutral particles moving in such changing underground structure could improve EM fields. Secondly because a charged particle is in someway carrying its own EM field on the back.
In my own research I could bring some elements concerning the first point at two moments: a) the partial derivation along the time of the Poynting's vector (calculated for a plane wave "moving" in vacuum) can be written in such a way that makes particularly clear the dependence of EM forces connected to a non vanishing connection; b) constructing a Riemann's curvature tensor inside an approach including variations of the second order of the local tetrad yields a relation of coherence that allows to accept the idea that first partial derivations of the basis vector generate fields mathematically isomorphic to an EM field. In both cases we got the proof or the conviction that variations of the local tetrad generate EM forces or fields.

Never mind, we stay with: ⌂$_{\blacktriangledown\Gamma}$ (**u**, **u**) = F. **u** – d**u**/ds (The LEL at p) which is exactly what* we pretend above. When the LEL is realized in p, then the main result of the split is represented by an element in $M_4(K)$: F, the Maxwell Faraday tensor; and the "rest" of the split is represented by an element of (E_4, K): – d**u**/ds, the relative acceleration of the particle at p. This interpretation was motivating a part of my work concerning what I did call the trivial split (per definition: split without rest; like a division can be), corresponding to particles with no acceleration; so concerning particles that appeared to be locally inertial one. For this particles, F must be a trivial split of the gravitational part of the LEL; an extensive investigation of this case has been done in etf68.pdf (French language).

The argumentation contra my interpretation will certainly evocate that the split #is entirely determinate by the characteristics of the particle (it is charged and F represents its own EM field in p) or by the circumstances (existence of an extern EM field acting in p). My interpretation does not reject these cases but is suggesting one more possibility: the split, when it is a priori realized, generates an EM field and is accompanied with an acceleration for which we have no indication if there is any logical dependence between, on one side: the topology and the particle in p, and the pair (F, – d**u**/ds) on the other side. Complicated interactions between the fields, as explained before, make the prevision not so automatic and easy. Thus we need a tool to help us in this prediction and give us the insurance that we shall at least find the classical and accepted situations again#.

The tool that I propose is this so-called equivalent scalar and it is defined by: S = **u.** {⌂$_{\blacktriangledown\Gamma}$ (**u**, **u**) **-** F. **u** + d**u**/ds}.

2. Scalar associated with a split and local energy.

Because I intuitively believe that a particular split must be related to the local energetic situation, I introduce the other original idea that the scalar S could be strongly connected to the Laplacian of a function h: $(E_4, K) \rightarrow K$ itself strongly related to the local potential of gravitation. That is why I also propose to look for the validity of situations where $S = \Delta h(\mathbf{x})$. The exploration of this idea has been done extensively in etf31v4.pdf in French language. I propose here a very short translation of this work.

2.1. Proposition:

There exists a priori one C^2 function h: $(E_4, K) \rightarrow K$ and one scalar T built with a primordial part of the Laplacian of h so that one can find a relatively simple relation like T[f(h)] = S. In extenso: T is a function of h.

2.1.1. Demonstration:

A) Let us consider the Hessian of this function of which we presuppose the existence. And let us built the scalar:

$$T = < d\mathbf{x} \,|.\, \mathrm{Hess}(h(\mathbf{x})) \,|\, d\mathbf{x} >$$

or in extenso :

$$T = \sum_\mu \sum_\nu [\partial_{\mu\nu}{}^2 h(\mathbf{x}) - \Gamma_{\mu\nu}{}^{\lambda}.\, \partial_\lambda h(\mathbf{x})].\, dx^\mu.\, dx^\nu.$$

A Taylor – Mac Laurin development until the first order of the ordinary derivation of the first partial derivates of the function h in a $(\ldots, x^\lambda, \ldots)$ coordinates system yields:

$$\forall\ \mu = 0, 1, 2, 3 : d(\partial_\mu h(\mathbf{x})) = \sum_\nu \partial^2{}_{\mu\nu} h(\mathbf{x}).\, dx^\nu + 0(3).$$

It follows:

$$\sum_\mu d(\partial_\mu h(\mathbf{x})).\, dx^\mu = \sum_\mu \sum_\nu \partial^2{}_{\mu\nu} h(\mathbf{x}).\, dx^\nu.\, dx^\mu + 0(3)$$

and, consequently:

$$\sum_\mu \sum_\nu \partial^2{}_{\mu\nu} h(\mathbf{x}).\, dx^\nu.\, dx^\mu = \{T + \sum_\mu \sum_\nu \Gamma_{\mu\nu}{}^{\lambda}.\, \partial_\lambda h(\mathbf{x}).\, dx^\mu.\, dx^\nu\} = \sum_\mu d(\partial_\mu h(\mathbf{x})).\, dx^\mu - 0(3)$$

So that:

$$T = \sum_\mu d(\partial_\mu h(\mathbf{x})).\, dx^\mu - \sum_\mu \sum_\nu \Gamma_{\mu\nu}{}^{\lambda}.\, \partial_\lambda h(\mathbf{x}).\, dx^\mu.\, dx^\nu - 0(3)$$

B) Let us now calculate explicitly S. In a first step and for future purposes, let us introduce a normalized 4-speed vector $\boldsymbol{\vartheta} = \mathbf{u}/c$ of which the norm is automatically contained in the interval [0, 1] since $c \geq |\mathbf{u}|$.

$$S/c^3 = \boldsymbol{\vartheta}.\, \{\triangle_{\blacktriangledown\Gamma}(\boldsymbol{\vartheta}, \boldsymbol{\vartheta}) - (1/c^2).\, F.\, \boldsymbol{\vartheta} + (1/c^2).\, d\boldsymbol{\vartheta}/ds\}$$

Since one can always write: $\boldsymbol{\vartheta} = d\mathbf{x}/ds$, it follows a particular representation of S:

$$ds.\, S/c = d\mathbf{x}.\, \{(c^2/ds).\, \triangle_{\blacktriangledown\Gamma}(\boldsymbol{\vartheta}, d\mathbf{x}) - F.\, d\mathbf{x} + d\boldsymbol{\vartheta}\}$$

Let us now make the calculation explicitly:

$$ds.\, S/c = \sum_\beta \sum_\gamma \sum_\varepsilon ((c^2/ds).\, \sum_\alpha \Gamma_{\alpha\gamma}{}^{\varepsilon}.\, \vartheta^\alpha.\, g_{\beta\varepsilon} - g_{\gamma\varepsilon}.\, F_{\varepsilon\beta}).\, dx^\beta.\, dx^\gamma + \sum_\beta \sum_\gamma g_{\beta\gamma}.\, dx^\beta.\, d\vartheta^\gamma$$

The scalars T and ds. S/c are "comparable" if we find realistic circumstances for which the three following relations hold simultaneously:

$$T = ds.\, S/c$$

$$T_2(°)(\nabla^*, \nabla^* h(\mathbf{x})).\, \boldsymbol{\vartheta} \approx G.\, d\boldsymbol{\vartheta}/ds$$

$$- [\ldots \sum_\mu \Gamma_{\mu\nu}{}^{\lambda}.\, \partial_\lambda h(\mathbf{x}) \ldots] = G.\, \{(c^2/ds).\, {}_\Gamma\Phi(\boldsymbol{\vartheta}) - F\}$$

2.2. Discussion.

2.2.1. Remark

Two quadratic equations can be compared if there exists a common scalar of proportionality between their terms of different power. For the simplicity and because they are a lot of other parameters to adjust the relations, we have make the choice of a factor 1. If s is an affine parameter as it is within the GTR, for example $s = c. t + s_0$ (a kind of curvilinear coordinate), we get T = S. dt. So we have found a scalar T proportional to S and this scalar is built, per definition, with an important part of the Laplacian of the function h, namely with its Hessian.

2.2.2. Remark

The second relation is quite more difficult to understand. If we make the assumption that it is a veiled formulation of the relation **p** = [M]. $\boldsymbol{\vartheta}$ = **constant** where [M] is an element of $M_4(K)$ related to the masses, then we get: d[M]/ds. $\boldsymbol{\vartheta}$ + [M]. d$\boldsymbol{\vartheta}$/ds = **0**. An identification can be proposed:

$$T_2(°)(\nabla^*, \nabla^* h(\mathbf{x})) = k. d[M]/ds$$

$$- G = k. [M]$$

Does it make sense? Let us for example consider the second relation of coherence. It implicitly propose a link between the metric tensor (its representation in $M_4(K)$) and the matrix related to the masses. Let us multiply it by c^2 and accept roughly the idea that c^2 [M] must be connected with the energy-momentum tensor [T]: $- c^2. G \approx k. [T]$. At the end, let us consider circumstances for which $[T] \approx [Ricci]$. We get $-c^2. G \approx [Ricci]$. These circumstances exist in the literature [01] and are described under the label: "Einstein's metrics". Even if it is not really a demonstration, it is an indication concerning the realism of the proposed identification.

Note that if the proposed relations are valid, then we can write:

$$d[G]/dt = - c. T_2(°)(\nabla^*, \nabla^* h(\mathbf{x}))$$

Circumstances for which:

$$c. T_2(°)(\nabla^*, \nabla^* h(\mathbf{x})) = [Ricci]$$

are those for which we get (if s is an affine parameter):

$$d[G]/dt = - [Ricci]$$

The trace of this relation should yield a relation very close to the equation of the Ricci flow [01].

Remark 2.2.3.

The third relation of coherence connects EM fields and variations of the metric.

3. Commentaries

3.1. Where could the scalar S comes from?

The procedure might sounds a little bit crazy or at least strange. In a first and too rapid critic, one would reject it because it is supposing that the Lorenz Einstein Law (LEL) could be not exactly true or that the motion of a first and given particle could be under the influence of something else modifying the LEL.

But this first reaction would forget at least two things:
1) One can imagine the existence in vacuum (the cosmological one; not the quantum approach of this notion) of a see of waves (with a average volumetric density of energy equivalent to 10^{-29} kg/cubic meter) and correlatively: of a function describing the local distribution of the energy. (In a similar way that has been used to describe semi-conductor with the Boltzman's equality).
2) The quantum approach admits the spontaneous birth of particles. It is not forbidden to believe that the behavior of each of these particles is also represented by a LEL. Their existence is a natural explanation for a perturbation of the LEL describing a first and given particle moving in vacuum.

We thus have at least two good reasons to accept the eventuality for S (the scalar associated to a representation of the LEL) not to vanish. And so, two physical motivations to introduce S.

3.2. Another important hypothesis:

The other non explained hypothesis that was make is the following. I decided to correlate the non vanishing S to the non vanishing of a Hessian; and conversely. (At the beginning of my quest, I must give it, I thought it was possible to connect S with the Laplacian of the potential of gravitation. It would have been a great result of my approach if it would have work in that way; but it didn't!).

3.3. The three relations:

This investigation finally results in 3 relations. The first one tells us a comfortable story: the more the duration (dt) is long, the more (Toutes choses étant égales par ailleurs) S vanish. That is: the more the LEL of the first particle under consideration becomes true again after having been perturbed by any phenomenon.

I leave the second relation for the end because it receives a clear explanation via the third one. The second relation, if interpreted as a special formulation of the relation: 4-momentum (mass by speed) is invariant, tells us the circumstances for which this approach makes sense and yields the masses (energies) for these circumstances... .

The third relation has the advantage to explain what the h function is or probably could be (wave function of the particle).

3.4. Comparison with the Boltzman's equality:

Suppose there exists a function f(**r**, **k**, t) describing the distribution of the different states (energy); suppose it depends on the waves vectors **k**, on **r** and on the time t. An ordinary derivate along the time of this function yields:

$$df/dt = \nabla_{\mathbf{x}}{}^*f.\ (d\mathbf{r}/dt) + \nabla_{\mathbf{k}}{}^*f.\ (d\mathbf{k}/dt) + (\partial f/\partial t)$$

where the symbol $\nabla_{\mathbf{w}}{}^*f$ is the gradient of f by respect for the vector **w** (here **w** is successively the position vector **r** and the wave vector **k**).

Since we know that:

$$\text{Quantum mechanic: } \mathbf{p} = (h/2\pi).\ \mathbf{k} \rightarrow \text{the force } \mathbf{F} = d\mathbf{p}/dt = (h/2\pi).\ d\mathbf{k}/dt$$

$$ds = c.\ dt\ ;\ c\text{: speed of light in vacuum.}$$

$$\text{4D speed vector of the wave: } \mathbf{u} = d\mathbf{r}/ds$$

$$W\text{: energy; } \nabla_{\mathbf{k}}{}^*f = (\partial f/\partial W).\ \nabla_{\mathbf{k}}{}^*W = (\partial f/\partial W).\ (h/2\pi).\ (d\mathbf{r}/dt) = (\partial f/\partial W).\ (h.\ c/2\pi).\ \mathbf{u}$$

We get:

$$df/dt = c.\ \nabla_{\mathbf{x}}{}^*f.\ \mathbf{u} + c.\ (\partial f/\partial W).\ \mathbf{u}.\ \mathbf{F} + (\partial f/\partial t)$$

This expression contains three parts; each of them is corresponding to a precise explanation:

1) $\nabla_{\mathbf{x}}{}^*f.\ \mathbf{u}$ is corresponding to a diffusion;

2) $(\partial f/\partial W).\ \mathbf{u}.\ \mathbf{F}$ is corresponding to the influence of extern fields and

3) $(\partial f/\partial t)$ could correspond here to the interaction between different waves.

In the reference [02; Pages 522-525] is speaking about electrons and it has been made the hypothesis that df = 0. The $(\partial f/\partial t)$ term is correlated to a probability (to go from a state characterized by **k*** to a state characterized by **k**). The circumstances described in this book are given by:

$$c. \nabla_{\mathbf{x}}*f. \mathbf{u} + c. (\partial f/\partial W). \mathbf{u}. \mathbf{F} + (\partial f/\partial t) = 0 \text{ (Boltzman's equality)}$$

3.5 What does it mean?

Let us suppose that **F** = [F]. **u** + δ**f**, that is let us suppose that the extern influence on a wave is due to an EM field ([F]. **u**) and to something else (δ**f**). Theoretically, this should result in: D**u**/ds = d**u**/ds + gravitational term = **F** = [F]. **u** + δ**f**. Note that what is called the LEL in the literature only is corresponding to the case where δ**f** = **0**. Coming back to the scalar S that I try to introduce in the discussion:

$$c. \nabla_{\mathbf{x}}*f. \mathbf{u} + c. (\partial f/\partial W). \mathbf{u}. [F]. \mathbf{u} - c. (\partial f/\partial W). \mathbf{u}. [F]. \mathbf{u} + c. (\partial f/\partial W). \mathbf{u}. \mathbf{F} + (\partial f/\partial t) = 0$$

$$c. \nabla_{\mathbf{x}}*f. \mathbf{u} + c. (\partial f/\partial W). \mathbf{u}. [F]. \mathbf{u} + c. (\partial f/\partial W). \mathbf{u}. \delta\mathbf{f} + (\partial f/\partial t) = 0$$

$$c. \nabla_{\mathbf{x}}*f. \mathbf{u} + c. (\partial f/\partial W). \mathbf{u}. [F]. \mathbf{u} + c. (\partial f/\partial W). S + (\partial f/\partial t) = 0$$

$$\nabla_{\mathbf{x}}*W. \mathbf{u} + \mathbf{u}. [F]. \mathbf{u} + S + (1/c). (\partial W/\partial t) = 0$$

Obviously, S owns the units of a power. If the idea to correlate this scalar S with a Hessian makes sense (as developed above), then we get the first relation of coherence:

$$T = d\mathbf{r}. \text{Hess}(h). d\mathbf{r} = S. dt.$$

The scalar T that can be now understood as a kind of projection of the Hessian of h has consequently the units of an energy. We thus discover *a posteriori* that the scenario proposed above is connecting a certain modification of the Euclidian curvature (given by the Hessian) with the apparition of a certain amount of energy. With other words, I am making the "toy" hypothesis that temporary modifications of the energetic state (for example in vacuum), respecting the Heisenberg's principle of uncertainty, can be due to the birth of particles and correspond to a perturbation in the evolution of the local curvature.

The domain of validity of my toy model is given by the second relation of coherence; namely, the metric must be Einstein and the masses associated with this toy scenario are the eigenvalues of the matrix of the masses which is proportional to the representation of the metric tensor. Example given: at the Minkowskian *limit*, we get only two values (+ 1, - 1) corresponding a priori to a particle and its anti-particle (+ m, - m).

4. Bibliography

[01] Ricci Flow and the Poincaré Conjecture; John W. Morgan and Gang Tian; chapter 1: preliminaries from Riemannian geometry; arXiv:math/0607607v2 [Math.DG] 21 Mar 2007.

[02] "Physik; Moleküle und Festkörper; Horst Hänsel, Werner Neumann; Spektrum Akademischer Verlag, Heidelberg, Berlin, Oxford; 1996.